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Two-dimensional anisotropic spiral self-avoiding walks

A J Guttmann and K J Wallace

Department of Mathematics, Statistics and Computer Science, The University of Newcastle, NSW 2308, Australia

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Abstract. Two models of anisotropic spiral self-avoiding random walks recently proposed by Manna have been investigated by the method of exact series expansions. The number of such n-step walks, c_n, appears to behave like $c_n \sim \text{constant} \times \mu^n n^\beta \exp(\alpha\sqrt{n})$ where both μ, which is known exactly, and the constant factor are model dependent but $\alpha \approx 0.14$ and $\beta \approx 0.9$ appear to be model independent. The mean square end-to-end distance exponent $\nu = 0.855 \pm 0.02$ for both models.

1. Introduction

Recently Manna (1984) proposed two variants of the problem of self-avoiding walks (SAW) on the square lattice. These variants combine the features of the normal SAW and the spiral SAW, recently introduced by Privman (1983) and subsequently solved by Guttmann and Wormald (1984) and Blöte and Hilhorst (1984). In the square lattice spiral SAW, no step through $-\pi/2$ is permitted. That is, in addition to the self-avoiding constraint, no 'turns to the right' are permitted at any step. This additional constraint dramatically changes the critical behaviour from that of the normal SAW, in that the number of n-step spiral SAW, denoted by s_n, behaves like

$$s_n \sim c \exp[2\pi(n/3)^{1/2}]/n^{7/4}(1+O(n^{-1/2})) \tag{1.1}$$

where $c = \pi/(4\times 3^{5/4})$, while the mean square end-to-end distance $\langle R_n^2\rangle$ behaves like

$$\langle R_n^2\rangle \sim n \ln n. \tag{1.2}$$

Manna's models are a mixture of the ordinary and spiral SAW, in that the spiral constraint applies only to steps in the direction of the x axis. To be more precise, if the nth step is in the $+x$ or $-x$ direction, the $(n+1)$th step cannot be in the $-y$ or $+y$ direction, respectively. If the nth step is in the $\pm y$ direction, then in model A the $(n+1)$th step cannot be in the same (i.e. $\pm y$) direction, while in model B the $(n+1)$th step is unconstrained, apart from the global SAW constraint that applies to all steps.

These two models, and the spiral SAW model, do not appear to be of any physical significance, but are interesting as models which display 'different' critical behaviour. As such, they are illuminating in our study of the mechanism whereby the universality class of SAW models changes, which is the subject of a subsequent paper in this series (Guttmann 1986).

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2. Series derivation

Manna obtained 28 and 21 terms in the series expansion of the generating function for model A and B walks respectively, and a similar number of terms for the mean square end-to-end distance sequence. Adopting the usual notational conventions, we write the chain generating function as

$$C(x) = \sum_{n=0}^{\infty} c_n x^n \tag{2.1}$$

where c_n is the number of n-step walks, $c_n \sim \mu^n f(n)$ and $\lim_{n\to\infty} (1/n) \log f(n) = 0$. For ordinary SAW $f(n)$ is believed to behave like $f(n) \sim n^{(\gamma-1)}$, but all that has been proved (Hammersley and Welsh 1962) is that $f(n) = O(e^{\sqrt{n}})$. For spiral SAW, as we have seen, $f(n) \sim \exp[2\pi(n/3)^{1/2}]/n^{7/4}$.

The mean square end-to-end distance, denoted $\langle R_n^2 \rangle$, is known to behave as

$$\langle R_n^2 \rangle \sim A n^{2\nu} \tag{2.2}$$

where $\nu = \frac{3}{4}$ for two-dimensional SAW (Nienhuis 1982, 1984) while for spiral SAW $\langle R_n^2 \rangle \sim An \log n$ (Blöte and Hilhorst 1984).

Manna's brief analysis of his series coefficients *assumed* $f(n) \sim n^g$, and he found $\mu \approx 2.04$ (model B), $\mu \approx 1.63$ (model A), $\gamma = 1.613$ (B) and $\gamma = 1.535$ (A), $\nu = 0.828 \pm 0.001$ (B) and $\nu = 0.852 \pm 0.002$ (A). An alternative analysis assuming $\langle R_n^2 \rangle \sim N^{2\nu} \log N$ gave $\nu = 0.735 \pm 0.015$ (A) and $\nu = 0.723 \pm 0.009$ (B).

Subsequently Whittington (1985) gave a convincing but non-rigorous argument which showed that $\mu = \frac{1}{2}(1 + \sqrt{5})$ (model A) and $\mu = 2$ (model B). Given that Manna's estimates for μ are in error by about 2%, and that the error in μ normally compares to the error in γ in the ratio $1 : N$, where N is the number of coefficients, the apparent precision in γ and claimed precision in ν seem somewhat optimistic. In this paper we have extended Manna's series by 12 and 9 terms for models A and B respectively, and have performed a thorough series analysis. The series coefficients are shown in table 1. They were derived by a straightforward counting procedure on a VAX 11/780. The algorithm used was essentially a backtracking algorithm, comprising nested DO loops. Making maximum use of symmetry, the series obtained took 140 h of CPU time for model A and 309 h for model B.

3. Series analysis

Firstly, assuming a singularity of the conventional type as assumed by Manna, so that $C(x) \sim A(1 - \mu x)^{-\gamma}$, unbiased ratio and d log Padé methods gave estimates of μ above the exact value, but slowly approaching the exact value as the number of terms used in the approximation increased. Convergence was found to be slow and, in the case of the Padé approximants, more than usually erratic. Performing a biased analysis, using the exact values of μ, the estimates of γ were found to be steadily increasing with the order of the approximants. For both models, this behaviour suggested $\gamma > 2.6$, which is numerically a large value for a critical exponent. This immediately suggests the following observation: if the exponent is so large, we would expect the singularity to dominate the series behaviour, and hence that the usual methods would converge rapidly to the correct values of μ and γ. Why does this not happen? There are two possible answers. One is that there are strong confluent or competing singularities

Table 1. Series coefficients of chain generating function and mean square end-to-end distance.

	Model A			Model B		
n	c_n	$c_n\langle R_n^2\rangle$	$\langle R_n^2\rangle$	c_n	$c_n\langle R_n^2\rangle$	$\langle R_n^2\rangle$
1	4	4	1.000 0000	4	4	1.000 0000
2	8	20	2.500 0000	10	28	2.800 0000
3	16	72	4.500 0000	24	120	5.000 0000
4	28	212	7.571 4286	54	424	7.851 8519
5	52	556	10.692 3077	124	1 340	10.806 4516
6	90	1 348	14.977 7778	272	3 944	14.500 0000
7	160	3 088	19.300 0000	608	11 024	18.131 5789
8	276	6 788	24.594 2029	1 314	29 664	22.575 3425
9	484	14 428	29.809 9174	2 884	77 444	26.852 9820
10	826	29 896	36.193 7046	6 178	197 428	31.956 6203
11	1 434	60 602	42.260 8089	13 388	493 324	36.848 2223
12	2 438	120 736	49.522 5595	28 486	1 212 616	42.568 8408
13	4 194	236 842	56.471 6261	61 168	2 938 432	48.038 7131
14	7 104	458 784	64.581 0811	129 446	7 034 908	54.346 2757
15	12 150	878 582	72.311 2757	276 020	16 662 788	60.368 0458
16	20 506	1 666 356	81.261 8746	581 572	39 102 224	67.235 3965
17	34 898	3 132 674	89.766 5769	1 233 204	90 996 020	73.788 2946
18	58 740	5 844 700	99.501 1917	2 588 906	210 202 724	81.193 6486
19	99 568	10 828 048	108.750 2812	5 464 816	482 319 552	88.259 0653
20	167 186	19 937 200	119.251 6120	11 437 088	1 100 067 208	96.184 2042
21	282 468	36 499 324	129.215 7837	24 050 760	2 495 186 664	103.746 6868
22	473 318	66 480 204	140.455 6852	50 201 640	5 631 375 824	112.175 1366
23	797 462	120 510 542	151.117 5981	105 228 216	12 650 695 032	120.221 5101
24	1 333 866	217 518 408	163.073 6581	219 139 194	28 299 422 184	129.139 0265
25	2 241 980	391 031 596	174.413 5077	458 067 944	63 056 390 232	137.657 2866
26	3 744 048	700 387 284	187.066 8549	951 999 224	139 992 817 520	147.051 3988
27	6 279 996	1 250 144 716	199.067 7567	1 985 163 932	309 747 043 484	156.030 9647
28	10 472 560	2 224 377 820	212.400 5802	4 118 332 532	683 191 279 168	165.890 2660
29	17 533 852	3 945 955 220	225.047 8229	8 569 510 852	1 502 421 404 196	175.321 7226
30	29 202 420	6 980 657 120	239.043 7888	17 749 322 414	3 294 913 250 516	185.636 0020
31	48 813 440	12 316 788 216	252.323 7087			
32	81 204 864	21 679 127 304	266.968 3346			
33	135 541 920	38 069 421 816	280.868 2496			
34	225 249 074	66 707 096 612	296.148 1503			
35	375 481 028	116 645 701 748	310.656 7125			
36	623 395 676	203 575 691 500	326.559 3576			
37	1 037 947 386	354 631 394 618	341.666 0607			
38	1 721 755 690	616 698 560 604	358.180 0625			
39	2 863 621 286	1 070 636 400 382	373.874 9972			
40	4 746 373 644	1 855 783 566 676	390.989 7757			

affecting the rate of convergence. Another more radical explanation is that the singularity is not of the assumed form.

Investigating the first possibility, we have used two methods. Firstly, the Baker-Hunter transformation (Baker and Hunter 1973), in which the series is so transformed that the poles of the Padé approximants give estimates of the dominant and sub-dominant exponents, provided the critical point is exactly known—as it is. Secondly, we have used the method of integral approximants, introduced by Guttmann and Joyce (1972) as the recurrence relation method. In this method, the series coefficients are fitted to a homogeneous or inhomogeneous linear ordinary differential equation whose derivatives have polynomial coefficients. From the theory of such equations, it follows that confluent singular behaviour can be well represented.

Both these methods gave no evidence of a conventional singularity structure with weaker confluent terms. Indeed, as the number of coefficients used in the representation increased, so did the estimate of γ, with the final estimates giving $\gamma > 3$, and with no evidence of a confluent exponent.

Such behaviour is reminiscent of that of spiral SAW, for which the first analyses gave $\gamma \approx 5.2$ (Privman 1983). The exact solution showed that the singularity was of the form $\exp(c\sqrt{n})$, corresponding to a value of infinity for the critical exponent.

The analysis of Hammersley and Welsh (1962) which proved that $c_n \sim \mu^n \times\exp(O(\sqrt{n}))$ for ordinary SAW can be repeated *mutatis mutandis* for the walks in question here, giving $f(n) \sim \exp(O\sqrt{n})$. Further, Whittington's analysis indicates more directly the connection between these walks and the number of paritions of the integers, which is known to behave like $\exp(O\sqrt{n})$.

We have therefore analysed the chain generating function under the assumption that

$$c_n \sim \mu^n \exp(\alpha\sqrt{n}) n^{\beta} \tag{3.1}$$

by analogy with the known result (1.1) for pure spiral SAW. In this case μ is known, so we are looking for estimates of α and β. These are found as follows. From (3.1) we form the sequence $\{t_n\}$ defined by

$$t_n = n^{1/2}[\log(d_n) - \log(d_{n-2})] \sim \alpha - 2\beta/n^{1/2} + O(1/n) \tag{3.2}$$

where $d_n = c_n/\mu^n$. Then the sequence $\{u_n\}$ defined by

$$u_n = t_n - t_{n-2} \sim 2\beta/n^{3/2} + O(1/n^2) \tag{3.3}$$

gives estimates of β and α, via the sequences $\{\beta_n^{(1)}\}$ and $\{\alpha_n^{(1)}\}$ defined by

$$\beta_n^{(1)} = n^{3/2}u_n/2 = \beta + O(1/\sqrt{n}) \tag{3.4}$$

and

$$\alpha_n^{(1)} = t_n + 2\beta_n^{(1)}/n^{1/2} = \alpha + O(1/n). \tag{3.5}$$

Improved estimates $\{\beta_n^{(2)}\}$ and $\{\alpha_n^{(2)}\}$ are given by

$$\beta_n^{(2)} = [n^{1/2}\beta_n^{(1)} - (n-2)^{1/2}\beta_{n-2}^{(1)}]/[n^{1/2} - (n-2)^{1/2}] \sim \beta + O(1/n) \tag{3.6}$$

and

$$\alpha_n^{(2)} = [n\alpha_n^{(1)} - (n-2)\alpha_{n-2}^{(1)}]/2 \sim \alpha + O(1/n^{3/2}) \tag{3.7}$$

while further extrapolation, which is in principle possible, no longer improves the exponent estimates.

In the above analysis we used alternate terms to eliminate the oscillatory behaviour that successive terms would cause due to the loose-packed lattice structure. The results of these calculations are shown in table 2. From these data we estimate

$$\begin{aligned} &\alpha = 0.13 \pm 0.03 \qquad \beta = -0.8 \pm 0.2 \qquad \text{(model A)} \\ &\alpha = 0.15 \pm 0.03 \qquad \beta = -0.9 \pm 0.2 \qquad \text{(model B).} \end{aligned} \tag{3.8}$$

Table 2. Estimates of exponents α and β for model A and model B walks, under the assumption that $c_n \sim \mu^n \exp(\alpha\sqrt{n})n^{\beta}$ with μ known.

Model B

n	$\alpha_n^{(1)}$	$\alpha_n^{(2)}$	$\beta_n^{(1)}$	$\beta_n^{(2)}$
9	0.265 88	0.134 60	−0.368 27	−0.794 13
10	0.283 89	0.271 10	−0.359 26	−0.446 64
11	0.300 28	0.455 07	−0.320 92	0.127 71
12	0.267 09	0.183 14	−0.390 12	−0.713 47
13	0.292 68	0.250 89	−0.336 68	−0.517 53
14	0.266 47	0.262 69	−0.394 39	−0.447 61
15	0.279 00	0.190 08	−0.363 88	−0.730 65
16	0.263 63	0.243 82	−0.402 14	−0.514 52
17	0.271 11	0.211 91	−0.381 23	−0.649 82
18	0.257 47	0.208 17	−0.416 48	−0.652 71
19	0.264 26	0.206 04	−0.397 06	−0.673 88
20	0.252 73	0.210 07	−0.428 19	−0.644 80
21	0.256 96	0.187 61	−0.414 42	−0.752 85
22	0.248 20	0.202 86	−0.439 74	−0.676 24
23	0.251 04	0.188 92	−0.429 23	−0.747 48
24	0.243 15	0.187 61	−0.452 79	−0.746 37
25	0.245 80	0.185 54	−0.442 89	−0.763 71
26	0.238 71	0.185 49	−0.464 71	−0.756 73
27	0.240 58	0.175 38	−0.456 90	−0.813 92
28	0.234 64	0.181 63	−0.476 04	−0.776 14
29	0.235 97	0.173 67	−0.469 75	−0.822 91
30	0.230 53	0.172 97	−0.487 75	−0.821 32

Model A

n	$\alpha_n^{(1)}$	$\alpha_n^{(2)}$	$\beta_n^{(1)}$	$\beta_n^{(2)}$
19	0.236 03	0.443 05	−0.302 47	0.552 72
20	0.212 71	0.115 64	−0.359 98	−0.808 80
21	0.224 62	0.116 19	−0.328 50	−0.835 73
22	0.218 71	0.278 61	−0.347 70	−0.096 03
23	0.220 10	0.172 73	−0.339 83	−0.583 34
24	0.212 59	0.145 36	−0.363 00	−0.707 25
25	0.217 82	0.191 56	−0.346 16	−0.494 83
26	0.210 14	0.180 75	−0.369 85	−0.537 43
27	0.213 61	0.160 95	−0.357 45	−0.645 14
28	0.207 78	0.177 06	−0.376 61	−0.555 75
29	0.210 04	0.161 89	−0.367 38	−0.640 56
30	0.205 05	0.166 81	−0.384 49	−0.609 06
31	0.207 21	0.166 15	−0.375 61	−0.618 15
32	0.202 30	0.161 10	−0.392 60	−0.639 89
33	0.204 28	0.158 80	−0.384 31	−0.658 55
34	0.199 98	0.162 82	−0.399 70	−0.630 35
35	0.201 32	0.152 63	−0.393 28	−0.693 63
36	0.197 60	0.157 13	−0.407 12	−0.663 08
37	0.198 78	0.154 18	−0.401 27	−0.684 71
38	0.195 15	0.151 12	−0.414 90	−0.698 75
39	0.196 30	0.150 39	−0.409 22	−0.707 57
40	0.192 98	0.151 74	−0.421 99	−0.695 03

It seems likely that α and β for the two models are the same, in which case the quotient

$$r_n = (c_n/\mu^n)_{\text{model A}}/(c_n/\mu^n)_{\text{model B}} \tag{3.9}$$

would approach a constant, the ratio of the two amplitudes. The sequence $\{r_n\}$ was formed and analysed, with the result that r_n appeared to be approaching a limit of around 0.5, but convergence was insufficient to assert this too strongly.

We next turn to the mean squared end-to-end distance series in order to estimate ν. We first establish that ν takes the same value for both model A and model B as follows. If $\langle R_n^2\rangle_A \sim n^{2\nu(A)}$ and $\langle R_n^2\rangle_B \sim n^{2\nu(B)}$, then $S_n = \langle R_n^2\rangle_A/\langle R_n^2\rangle_B \sim n^{2(\nu(A)-\nu(B))} = n^{2\phi}$. A straightforward analysis of the sequence $\{S_n\}$ paralleling that reported in Guttmann and Torrie (1985) gives the estimate $|\phi| < 0.01$ from which we deduce that $\nu(A)$ is most likely equal to $\nu(B)$. In the following, we assume this to be true.

The next difficulty encountered is that, in analogy with spiral self-avoiding walks, there may be a confluent logarithmic exponent. As Manna's analysis implies, it is difficult to distinguish a singularity of the form $n^{0.85}$ from a singularity of the form

Table 3. Estimates of ν assuming $\langle R_n^2\rangle \sim n^{2\nu}(c_0 + c_1/n + c_2/n^2 + \ldots)$. $\nu_n^{(1)}$ assumes $c_n = 0$ for $n \geq 1$, $\nu_n^{(2)}$ assumes $c_n = 0$ for $n \geq 2$, $\nu_n^{(3)}$ assumes $c_n = 0$ for $n \geq 3$.

	Model A			Model B		
n	$\nu_n^{(1)}$	$\nu_n^{(2)}$	$\nu_n^{(3)}$	$\nu_n^{(1)}$	$\nu_n^{(2)}$	$\nu_n^{(3)}$
10	0.865 75	0.897 86	0.505 55	0.778 69	0.815 70	0.828 45
11	0.869 63	0.914 23	1.562 31	0.788 43	0.820 37	0.812 16
12	0.859 86	0.798 15	0.299 61	0.786 37	0.824 73	0.845 25
13	0.867 62	0.844 51	0.461 05	0.793 75	0.823 03	0.829 73
14	0.861 15	0.877 19	1.351 47	0.792 26	0.827 59	0.835 52
15	0.863 87	0.813 38	0.611 02	0.798 22	0.827 27	0.840 06
16	0.860 30	0.848 08	0.644 32	0.796 91	0.829 45	0.835 51
17	0.863 80	0.862 73	1.232 89	0.801 91	0.829 60	0.837 80
18	0.859 60	0.847 99	0.847 21	0.800 78	0.831 80	0.840 66
19	0.862 40	0.837 83	0.626 16	0.805 01	0.831 36	0.838 43
20	0.859 27	0.853 13	0.899 45	0.804 04	0.833 35	0.839 96
21	0.861 43	0.842 54	0.887 30	0.807 71	0.833 28	0.841 92
22	0.858 55	0.843 74	0.749 84	0.806 82	0.834 62	0.840 64
23	0.860 57	0.842 04	0.836 81	0.810 05	0.834 69	0.841 79
24	0.857 99	0.845 50	0.864 87	0.809 25	0.836 00	0.843 30
25	0.859 73	0.839 97	0.816 19	0.812 12	0.835 86	0.842 32
26	0.857 44	0.844 01	0.826 12	0.811 40	0.837 13	0.843 59
27	0.858 98	0.840 01	0.840 43	0.813 97	0.837 07	0.844 33
28	0.856 91	0.842 83	0.827 49	0.813 30	0.838 08	0.844 06
29	0.858 31	0.839 82	0.837 29	0.815 63	0.838 12	0.844 91
30	0.856 41	0.842 23	0.833 76	0.815 02	0.839 10	0.845 96
31	0.857 68	0.839 16	0.829 55			
32	0.855 95	0.841 76	0.834 78			
33	0.857 10	0.838 90	0.834 97			
34	0.855 51	0.841 24	0.832 92			
35	0.856 58	0.838 97	0.840 07			
36	0.855 10	0.840 84	0.834 07			
37	0.856 09	0.838 71	0.834 21			
38	0.854 71	0.840 80	0.840 06			
39	0.855 64	0.838 69	0.838 27			
40	0.854 36	0.840 59	0.836 48			

$n^{0.72}\log(n)$ by elementary methods. However we can rule out the confluent logarithmic term by the following analysis.

If $\langle R_n^2\rangle \sim An^{2\nu}(1+cn^{-\Delta})$ then the sequence

$$\nu(n)=\tfrac{1}{2}\ln(\langle R_n^2\rangle/\langle R_{n-2}^2\rangle)/\ln(n/(n-2))\sim \nu+\mathrm{O}(1/n^{\Delta}). \tag{3.10}$$

If however $\langle R_n^2\rangle \sim An^{2\nu}(\log(n))^{\alpha}$, then the sequence

$$\nu(n)\sim \nu+(\alpha/2)\log n \tag{3.11}$$

that is, $\nu(n)$ approaches ν *from above* if $\alpha>0$. For model B we find $\nu(n)$ is a pairwise monotone *increasing* function for all n (see table 3) so that the limit is being approached from below. If $\langle R_n^2\rangle \sim An^{2\nu}(\log n)^{\alpha}$ with $\alpha<0$, then $\nu(n)$ defined by (3.10) approaches ν from below. If it is accepted that both series have the same exponent ν and the same confluent logarithmic correction (if any) then the behaviour of model A extrapolants, which approach ν from above, imply $\alpha>0$. This contradiction implies that neither series has a confluent logarithmic term.

Table 4. Estimates of ν assuming $\langle R_n^2\rangle \sim n^{2\nu}(b_0+b_1/n^{1/2}+b_2/n+b_3/n^{3/2}+\ldots)$. $\nu_n^{(1)}$ assumes $b_n=0$ for $n\geq 1$, $\nu_n^{(2)}$ assumes $b_n=0$ for $n\geq 2$, $\nu_n^{(3)}$ assumes $b_n=0$ for $n\geq 3$.

	Model A			Model B		
n	$\nu_n^{(1)}$	$\nu_n^{(2)}$	$\nu_n^{(3)}$	$\nu_n^{(1)}$	$\nu_n^{(2)}$	$\nu_n^{(3)}$
10	0.865 75	0.880 91	0.804 06	0.778 69	0.857 08	0.870 75
11	0.869 63	0.890 81	1.033 12	0.788 43	0.855 67	0.776 48
12	0.859 86	0.830 41	0.715 64	0.786 37	0.866 75	0.915 14
13	0.867 62	0.856 54	0.770 16	0.793 75	0.854 85	0.850 32
14	0.861 15	0.868 86	0.975 34	0.792 26	0.865 76	0.859 81
15	0.863 87	0.839 53	0.788 17	0.798 22	0.858 46	0.881 95
16	0.860 30	0.854 40	0.807 15	0.796 91	0.864 24	0.853 59
17	0.863 80	0.863 28	0.946 80	0.801 91	0.859 07	0.863 64
18	0.859 60	0.853 96	0.852 33	0.800 78	0.864 70	0.868 41
19	0.862 40	0.850 46	0.798 96	0.805 01	0.859 21	0.860 40
20	0.859 27	0.856 28	0.866 16	0.804 04	0.864 25	0.860 19
21	0.861 43	0.852 22	0.860 19	0.807 71	0.860 16	0.869 14
22	0.858 55	0.851 32	0.827 70	0.806 82	0.863 77	0.858 97
23	0.860 57	0.851 52	0.847 98	0.810 05	0.860 48	0.863 86
24	0.857 99	0.851 88	0.854 84	0.809 25	0.863 94	0.865 85
25	0.859 73	0.850 06	0.842 01	0.812 12	0.860 62	0.862 25
26	0.857 44	0.850 86	0.844 98	0.811 40	0.863 91	0.863 44
27	0.858 98	0.849 68	0.847 40	0.813 97	0.861 08	0.866 87
28	0.856 91	0.850 00	0.844 62	0.813 30	0.863 80	0.862 37
29	0.858 31	0.849 23	0.846 32	0.815 63	0.861 42	0.865 94
30	0.856 41	0.849 44	0.845 66	0.815 02	0.864 02	0.867 12
31	0.857 68	0.848 57	0.843 98			
32	0.855 95	0.848 97	0.845 53			
33	0.857 10	0.848 15	0.844 93			
34	0.855 51	0.848 48	0.844 71			
35	0.856 58	0.847 90	0.845 96			
36	0.855 10	0.848 07	0.844 67			
37	0.856 09	0.847 52	0.844 27			
38	0.854 71	0.847 85	0.845 94			
39	0.855 64	0.847 27	0.845 05			
40	0.854 36	0.847 56	0.844 85			

We have therefore analysed for the form $\langle R_n^2\rangle \sim An^{2\nu}$. Two methods of analysis were used. In the first, we assumed that

$$\langle R_n^2\rangle \sim n^{2\nu}(c_0 + c_1/n + c_2/n^2 + c_3/n^3 + \ldots) \tag{3.12}$$

and successively eliminated c_0, c_1, c_2 by forming a Neville table to the sequence $\nu(n)$ defined above. The second method of analysis was suggested by the form (3.1) found for the chain generating function for this model and for the spiral SAW model. In both cases correction terms of order $n^{1/2}$ appeared, and such correction terms will impress themselves on the $\langle R_n^2\rangle$ sequence, so the second method of analysis assumed

$$\langle R_n^2\rangle \sim n^{2\nu}(b_0 + b_1/n^{1/2} + b_2/n + b_3/n^{3/2} + \ldots) \tag{3.13}$$

with successive estimates eliminating b_0, b_1, b_2, etc.

The results for both methods applied to both models are shown in tables 3 and 4. The first method of analysis gave $\nu \approx 0.844$, while the second gave $\nu \approx 0.864$. Subsequently we tried further refinements of the analysis which included repeatedly averaging successive terms in the $\langle R_n^2\rangle$ sequence, which has the effect of leaving the dominant exponent unchanged while reducing the effect of the singularity on the negative real axis (at -1). The results of these analyses (not shown) confirmed the above results. Further, we obtained similar estimates for model A from both methods of analysis. We thus conclude that $\nu = 0.855 \pm 0.020$ for both models.

4. Conclusion

The two models discussed here appear to display quite different behaviour from that found by any other self-avoiding walk model. The apparent form of c_n includes features of both the ordinary SAW model and the spiral SAW model.

The critical exponent for the mean square end-to-end distance sequence also takes on a value not shared by any other SAW type model. The initially surprising fact that ν is larger than the corresponding value for ordinary or spiral SAW simply means that these two constraints combine to cause the walk to spread out more than would be the case for each constraint individually.

We find strong, though not overwhelming, evidence that the two models display the same critical behaviour.

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